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ON POLARIZED SURFACES (X, L) WITH $h^0(L) > 0$, $\kappa(X) = 2$, AND $g(L) = q(X)$

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ABSTRACT. Let X be a smooth projective surface over $\mathbb{C}$ and L an ample Cartier divisor on X. If the Kodaira dimension $\kappa(X) \leq 1$ or $\dim H^0(L) > 0$, the author proved $g(L) \geq q(X)$, where $q(X) = \dim H^1(\mathcal{O}_X)$. If $\kappa(X) \leq 1$, then the author studied (X, L) with $g(L) = q(X)$. In this paper, we study the polarized surface (X, L) with $\kappa(X) = 2$, $g(L) = q(X)$, and $\dim H^0(L) > 0$.

0. Introduction

Let X be a smooth projective variety over the complex number field with $\dim X = n$ and L a Cartier divisor on X. The pair (X, L) is called a polarized (resp. quasi-polarized) manifold if L is ample (resp. nef-big). The sectional genus is defined by the following formula ([Fj1]):

$$g(L) = 1 + \frac{1}{2}(K_X + (n-1)L)L^{n-1}.$$

Then there is the following conjecture.

Conjecture (p.111 in [Fj1]). *Let (X, L) be a quasi-polarized manifold. Then $g(L) \geq q(X)$, where $q(X) = \dim H^1(\mathcal{O}_X)$.*

It is known that this conjecture is true if one of the following cases hold :

(1) L is spanned.
(2) $\dim X = 2$, and $\kappa(X) \leq 1$. (See [Fk1])
(3) $\dim X \geq 3$, $L^n \geq 2$ and $\kappa(X) = 0, 1$. (See [Fk2])
(4) $\dim X = 2$, and $h^0(L) > 0$.

It is natural that we study (X, L) with $g(L) = q(X)$ when the above conjecture is true. If $g(L) = q(X)$ and L is ample and spanned, (X, L) is one of the following types. (See [So] [SV])

(1): (X, L) is a scroll over a smooth curve. (That is, there is a smooth curve C and a surjective morphism $f : X \to C$ with connected fibers such that any fiber F of f is $\mathbb{P}^{n-1}$ and $L_F = \mathcal{O}(1)$.)

(2): $\Delta(L) = 0$, where $\Delta(L)$ is Δ-genus, i.e. $\Delta(L) = n + L^n - h^0(L)$ (see [Fj1], [Fj2]).

If (X, L) is an L-minimal quasi-polarized surface with $g(L) = q(X)$ and $\kappa(X) \leq 1$ (for the definition of "L-minimal", see Definition 1.10), then (X, L) is one of the following types (see [Fk1]).

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(1) The case in which $\kappa(X) = -\infty$.
(1-1): $(X, L) = (\mathbb{P}^2, \mathcal{O}(r))$, $r = 1$ or 2
(1-2): $(X, L) = (\mathbb{P}^1\text{-bundle}, L)$, $L|_{\text{fiber}} = \mathcal{O}(1)$
(2) The case in which $\kappa(X) = 0$.
(2-1): $(X, L) = (J(C), L)$, (where $J(C)$ is the jacobian variety of a smooth curve C of genus 2, and L is the translation class of C.)
(2-2): $(X, L) = (C_1 \times C_2, F_1 + F_2)$, (where C_k is an elliptic curve and F_k is a fiber of $C_1 \times C_2 \to C_k$ ($k = 1$ or 2).)
(2-1)$'$: X is one point blowing up of (2-1), and $L.E = 1$ for the (-1)-curve E.
(2-2)$'$: X is one point blowing up of (2-2), and $L.E = 1$ for the (-1)-curve E.
(3) The case in which $\kappa(X) = 1$.
$(X, L) = (F \times C, L), L \equiv F + C$, (where F is an elliptic curve and C is a smooth curve of genus $g(C) \geq 2$.)
(3)$'$: X is one point blowing up of (3), and $L.E = 1$ for the (-1)-curve E.

In this paper we study the case in which (X, L) is a polarized surface with $\kappa(X) = 2$, $h^0(L) > 0$, and $g(L) = q(X)$. Main result is the following.

Theorem 4.2. *Let (X, L) be a polarized surface with $\kappa(X) = 2$ and $h^0(L) > 0$. If $g(L) = q(X)$, then $h^0(L) = 1$ and $1 \leq L^2 \leq 4$.*

Let D be the effective divisor which is linearly equivalent to L. Then D is a reduced divisor and is one of the following types.

(1) D is an irreducible reduced smooth curve.

(2) $X \cong C_1 \times C_2$ and $D = F_1 + F_2$, where F_i is a fiber of the projection $X \to C_i$ for $i = 1, 2$. In particular $L^2 = 2$.

We shall study polarized surfaces (X, L) with $h^0(L) > 0$, $\kappa(X) \geq 0$, and $g(L) = q(X) + 1$ in a forthcoming paper.

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1. Preliminaries

Definition 1.1. Let D be a Cartier divisor on a smooth projective variety X. Then D is called pseudo effective if $\kappa(mD + H) \geq 0$ for all big divisors H and all natural numbers m.

Remark 1.2. D is pseudo effective if and only if there is a big Cartier divisor H such that $\kappa(mD + H) \geq 0$ for all natural numbers m. (For a proof, see [Mo] p.318) We remark that D is pseudo effective if and only if there is a big Cartier divisor H such that $\kappa(mD + H) \geq 0$ for any sufficiently large natural number m.

Lemma 1.3 (Kodaira-Ramanujam-Bombieri-Catanese). *Let X be a smooth projective surface with $q(X) \geq 1$ and D an effective divisor on X. We put*

$$\alpha(D) = \dim \ker(H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D)).$$

Then

(1) *If* $\alpha(D) = q(X)$*, then* D *is contracted by the Albanese map* $a : X \to \mathrm{Alb}(X) = A$.

(2) *If* $0 < \alpha(D) < q(X)$*, then there is an Abelian variety* G *with* $\dim G > 0$ *and a morphism* $f : X \to G$ *such that* $f(X)$ *is not a point and* D *is contracted by* f.

Proof (cf. Remark 6.8 in [Ca], p. 48 Remark in [Ra]). By Lemma 6 in [Ra], $\alpha(D) = \alpha(D_{\mathrm{red}})$. Hence we may assume that D is reduced. Let B be the Abelian subvariety of A generated by $a(x) - a(y)$ where x and y belong to the same connected component of D. Let $u : B \to A$. Then $\alpha(D) = \dim\ker(\mathrm{Pic}^0 A \to \mathrm{Pic}^0 B)$ by p. 48 Remark in [Ra].

(1) The case in which $q(X) = \alpha(D)$.

Then $\hat{u} : \mathrm{Pic}^0 A \to \mathrm{Pic}^0 B$ is 0-map. (We denote the dual by $\hat{}$ and we say that a homomorphism $f : A_1 \to A_2$ of Abelian varieties A_1 and A_2 is 0-map if $f(A_1) = 0$.) Here a natural homomorphism $u : B \to A$ is 0-map by duality. Hence $B = 0$. By construction of B, $a(x) - a(y) = 0$ for x and y belonging to the same connected component of D. Therefore $a(D)$ are points.

(2) The case in which $0 < \alpha(D) < q(X)$.

Let G' be the connected component of the kernel of $\hat{u} : \mathrm{Pic}^0 A \to \mathrm{Pic}^0 B$ which contains the identity of $\mathrm{Pic}^0 A$. Then G' is an Abelian variety with $\dim G' > 0$ and let $v : G' \to \mathrm{Pic}^0 A$. Then $\hat{u} \circ v$ is 0-map. By taking its dual, $h : B \to G$ is 0-map (where G is the dual of G'). On the other hand, $a(x) - a(y) \in B$ where x and y belong to the same connected component of D. Hence $h(a(x) - a(y)) = 0$. We put $f = \hat{v} \circ a$. For any x and y which belong to the same connected component of D, $f(x) - f(y) = \hat{v} \circ a(x) - \hat{v} \circ a(y) = \hat{v}(a(x) - a(y)) = h(a(x) - a(y)) = 0$. Then $f(D)$ are points.

Next we prove $f(X)$ is not a point. $A = \mathrm{Alb}(X)$ is generated by $a(X)$. Hence if $\hat{v} : A \to G$ is not 0-map, then $f(X)$ is not a point. If $\hat{v}$ is 0-map, then $v : G' \to \mathrm{Pic}^0 A$ is also 0-map. Hence $G' = 0$. But this is the contradiction by hypothesis. □

Lemma 1.4. *Let* (X, L) *be a quasi-polarized surface. Assume that* $L^2 \geq \frac{2b}{a} LF$ *where* $a, b \in \mathbb{N}$ *and* F *is an irreducible reduced curve with* $F^2 = 0$ *and* $LF > 0$.
Then $aL - bF$ *is pseudo effective.*

Proof. Let A be an ample divisor on X such that $(aL - bF)A > 0$. (The existence of A can be seen as follows. By assumption, F is nef. Since $(B + F)F = BF$, and $(B + F)L > LB$ for any ample divisor B, we have $(aL - bF)(B + F) > (aL - bF)B$. We put $A = B + nF$ for $n \gg 0$. Then A is ample and $(aL - bF)A > 0$.) We consider $t(aL - bF) + A$ for $t \in \mathbb{N}$. We prove that the Iitaka dimension of $t(aL - bF) + A$ is nonnegative for $t \gg 0$.

For $i, m \in \mathbb{N}$, there is an exact sequence

$$\begin{aligned} 0 &\to \mathcal{O}(mtaL + mA - iF) \to \mathcal{O}(mtaL + mA - (i-1)F) \\ &\to \mathcal{O}(mtaL_F + mA_F - (i-1)F_F) \to 0. \end{aligned}$$

Hence

$$\begin{aligned} &h^0(mtaL + mA - (i-1)F) \\ &\quad \leq h^0(mtaL + mA - iF) + h^0(mtaL_F + mA_F - (i-1)F_F). \end{aligned}$$

Therefore
(1)
$$h^0(mtaL+mA) \le \sum_{i=0}^{mtb-1} h^0(mtaL_F+mA_F-iF_F)+h^0(m(taL-tbF)+mA).$$

Next we calculate $h^0((mtaL+mA-iF)_F)$. Let $\mu : X' \to X$ be a birational morphism such that the strict transform F' of F is smooth on X'. Let $\mu_{F'}$ be the restriction of μ to F'. Then
$$\begin{aligned} h^0((mtaL+mA-iF)_F) &\le h^0((\mu_{F'})^*((mtaL+mA-iF)_F)) \\ &= h^0((\mu^*(mtaL+mA-iF))_{F'}). \end{aligned}$$

On the other hand, $\deg \mu^*(mtaL+mA-iF)_{F'} = mtaLF+mAF > 2g(F')-2$ for any $m > 2g(F')-2$. Hence $h^1(\mu^*(mtaL+mA-iF)_{F'})=0$ for any $m \gg 2g(F')-2$ and $i>0$. By the Riemann-Roch Theorem, we have
$$\begin{aligned} h^0((mtaL+mA-iF)_F) &\le h^0(\mu^*(mtaL+mA-iF)_{F'}) \\ &= 1-g(F')+m(taLF+AF). \end{aligned} \tag{2}$$

Therefore by (1) and (2), $h^0(mtaL+mA)-mtb(1-g(F'))-m^2tb(taLF+AF) \le h^0(m(taL-tbF)+mA)$. For $m \gg 0$,
$$h^0(mtaL+mA) = \frac{(taL+A)^2}{2}m^2 + (\text{lower degree of } m)$$
by the Riemann-Roch Theorem. Hence
$$\begin{aligned} &h^0(mtaL+mA)-mtb(1-g(F'))-m^2tb(taLF+AF) \\ &= (\frac{(taL+A)^2}{2}-tb(taLF+AF))m^2+(\text{lower degree of } m) \\ &= \frac{1}{2}(aL(aL-2bF)t^2+(2A(aL-bF))t+A^2)m^2+(\text{lower degree of } m). \end{aligned}$$

If $L^2 > \frac{2b}{a}LF$, then $h^0(m(taL-tbF)+mA)>0$ for $m\gg 0$, $t \gg 0$. Hence $aL-bF$ is pseudo effective. If $L^2 = \frac{2b}{a}LF$, then $h^0(m(taL-tbF)+mA)>0$ for $m \gg 0$ and $t \gg 0$ by the choice of A. Therefore $aL-bF$ is pseudo effective. □

Remark 1.4.1. We remark that in [De] the following lemma is proved: Let X be a projective algebraic manifold with $\dim X = n$, and let F and G be nef line bundles over X. If $F^n > nF^{n-1}G$, then $k(F-G)$ has a non trivial section for all large positive k. (See Lemma 4.1 in [De].)

Theorem 1.5 (Reider). *Let X be a smooth projective surface over $\mathbb{C}$ and L a nef divisor on X. If $L^2 \ge 5$ and $p \in \mathrm{Bs}\,|K_X+L|$, then there is an effective divisor $E \ni p$ such that*

(1) $LE=0$ *and* $E^2=-1$

or

(2) $LE=1$ *and* $E^2=0$.

Proof. See [Re]. □

Definition 1.6. Let X be a smooth projective surface over $\mathbb{C}$ and D an effective divisor on X. Then D is called 1-connected if $D_1D_2>0$ for any $D=D_1+D_2$, $D_1>0$, $D_2>0$.

Remark 1.7. If D is a reduced connected effective divisor, then D is 1-connected. But in general, a connected effective divisor is not always 1-connected.

Lemma 1.8 (Ramanujam [Ra]). *Let X be a smooth projective surface over $\mathbb{C}$ and D be a nef and big effective divisor. Then D is 1-connected.*

Lemma 1.9. *Let (X, L) be a quasi-polarized surface with $\kappa(X) = 2$ and $g(L) = q(X)$. Then $q(X) \geq 2$. In particular, $p_g \geq 2$.*

Proof. Since $\kappa(X) = 2$, we have $p_g \geq q(X) = g(L) \geq 2$. □

Definition 1.10. Let (X, L) be a quasi-polarized surface. Then (X, L) is called L-minimal if $LE > 0$ for any (-1)-curve E on X. Let C be a smooth curve and $f : X \to C$ a surjective morphism with connected fibers. Then (f, X, C, L) is called a quasi-polarized fiber space if L is nef and big. (f, X, C, L) is called relatively L-minimal if $LE > 0$ for any (-1)-curve E on X such that $f(E)$ is a point.

Lemma 1.11. *Let (X, L) be an L-minimal quasi-polarized surface with $\kappa(X) \geq 0$. Then $K_X + L$ is nef.*

Proof. Assume that $K_X + L$ is not nef. Then there is a (-1)-curve E on X such that $(K_X + L)E < 0$. Hence $LE = 0$. But this is a contradiction. □

Lemma 1.12. *Let (f, X, C, L) be relatively L-minimal quasi-polarized fiber space with $\kappa(X) \geq 0$. Then $K_{X/C} + L$ is nef, where $K_{X/C} = K_X - f^*K_C$ is the relative canonical bundle.*

Proof. If $K_{X/C} + L$ is not f-nef, then there is a (-1)-curve E on X such that $f(E)$ is a point and $(K_{X/C} + L)E = (K_X + L)E < 0$. Since $K_X E = -1$, $LE = 0$. But this is a contradiction. Hence $K_{X/C} + L$ is f-nef. Let $\mu : X \to X'$ be the relatively minimal model of $f : X \to C$. Then we have a surjective morphism $f' : X' \to C$ with connected fibers such that $f = f' \circ \mu$. If an irreducible curve D on X is not contained in a fiber of f, then $\mu(D) = D'$ is a curve and $K_{X/C}D \geq K_{X'/C}D'$. On the other hand, $K_{X'/C}$ is nef by Arakelov's theorem. Hence $K_{X/C}D \geq 0$. Therefore $K_{X/C} + L$ is nef. □

Definition 1.13. Let D be an effective divisor on X. Then the dual graph $G(D)$ of D is defined as follows.

1. The vertices of $G(D)$ correspond to irreducible components of D.
2. For any two vertices v_1 and v_2 of $G(D)$, the number of edges joining v_1 and v_2 equals $\sharp\{B_1 \cap B_2\}$, where B_i is the component of D corresponding to v_i for $i = 1, 2$.

Let C_i be an irreducible component of D. If the degree of the vertex corresponding to C_i is 1, we say that C_i is a tip curve of D.

2. $L^2 \geq 5$ case

Theorem 2.1. *Let (X, L) be a polarized surface over $\mathbb{C}$ with $\kappa(X) = 2$, $h^0(L) > 0$, and $L^2 \geq 5$. Then $g(L) \geq q(X) + 1$.*

Proof. Suppose that $g(L) = q(X)$. By Lemma 1.9, $p_g \geq 2$. Let $D = \sum_i a_i D_i$ be an effective divisor which is linearly equivalent to L. Since $g(L) = q(X)$, we have $h^0(K_X + L) = h^0(K_X)$ and $h^0(L) = 1$. Hence D is a fixed component of $|K_X + L|$. Since L is ample, for any $p \in D$, there is an effective divisor $E_p \ni p$ such that

$LE_p = 1$ and $E_p^2 = 0$ by Theorem 1.5. In particular E_p is an irreducible reduced curve.

Claim 2.2. *$LD_i \neq 1$ or $D_i^2 \neq 0$ for some i.*

Proof. Assume that $LD_i = 1$ and $D_i^2 = 0$ for any i. Let D_1 and D_2 be irreducible components of D such that $D_1D_2 > 0$. Then $L(D_1 + D_2) = 2$ and $(D_1 + D_2)^2 > 0$ by hypothesis. But by the Hodge index theorem this is a contradiction because $L^2 \geq 5$. □

By Claim 2.2, there exists an irreducible reduced curve B of a component of D and for any $p \in B$ there is an irreducible reduced curve E_p on X such that

(1) $E_p \ni p$,
(2) $LE_p = 1$,
(3) $E_p^2 = 0$,
(4) $E_p \neq B$.

We consider $\{E_p\}_{p\in B}$.

Claim 2.3. *$E_p \neq E_q$ for $p, q \in B$ such that $p \neq q$.*

Proof. If $E_p = E_q$, then $q \in E_p$. Therefore $E_pB \geq 2$. On the other hand, E_p is nef. Hence $(L - B)E_p \geq 0$. Therefore $LE_p \geq 2$. This is a contradiction. □

Claim 2.4. *E_p and E_q are disjoint for $p \neq q \in B$.*

Proof. If $E_pE_q > 0$, then $L(E_p + E_q) = 2$ and $(E_p + E_q)^2 > 0$. But by the Hodge index theorem this is impossible since $L^2 \geq 5$. □

We take an $E_p \in \{E_p\}_{p\in B}$. Let $\alpha(E_p) = \dim \mathrm{Ker}(H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{E_p}))$.

(1) The case in which $\alpha(E_p) = 0$.

In this case, $q(X) \leq g(E_p)$. Since $L^2 > 4LE_p$, $L - 2E_p$ is pseudo effective by Lemma 1.4. Therefore

$$\begin{aligned} g(L) &\geq 1 + \frac{1}{2}(K_X + L)(2E_p) = 2 + K_XE_p \\ &= 2g(E_p) \geq 2q(X). \end{aligned}$$

This is a contradiction because $g(L) = q(X)$ and $q(X) \geq 2$.

(2) The case in which $\alpha(E_p) = q(X)$.

Let $a : X \to \mathrm{Alb}(X) = A$ be the Albanese map of X. By Lemma 1.3, $a(E_p)$ is a point. On the other hand $a(X)$ is a curve since $E_p^2 = 0$. Hence $g(L) \geq q(X) + 1$ by Theorem 5.5 in [Fk1]. Therefore this case cannot occur.

(3) The case in which $0 < \alpha(E_p) < q(X)$.

By Lemma 1.3, there is an Abelian variety G with $\dim G > 0$ and a morphism $f : X \to G$ such that $f(X)$ is not a point and $f(E_p)$ is a point. Since $E_p^2 = 0$, $f(X)$ is a curve. By Stein factorization, there is a fiber space $h : X \to C$ (i.e. h is a surjective morphism with connected fibers and C is a smooth curve) with $g(C) \geq 1$ since G is an Abelian variety. We remark that E_p is contained in a fiber of h. Since $E_p^2 = 0$, m_pE_p is a fiber of h. On the other hand, for any $E_q \in \{E_p\}_{p\in B}$ such that $E_q \neq E_p$, E_q is contained in a fiber of h and m_qE_q is a fiber of h and $m_qE_q \neq m_pE_p$. Since $\sharp\{E_p\}_{p\in B}$ is infinitely many, $m_q = 1$ for a general $q \in B$.

Hence E_q is a fiber of h for a general $q \in B$. Since $L^2 - 5LE_q \geq 0$, $L - \frac{5}{2}E_q$ is pseudo effective by Lemma 1.4. Since $K_{X/C} + L$ is nef by Lemma 1.12, we have

$$\begin{aligned} g(L) &= g(C) + \frac{1}{2}(K_{X/C} + L)L + (LF - 1)(g(C) - 1) \\ &\geq g(C) + \frac{1}{2}(K_{X/C} + L)(\frac{5}{2}E_q) \\ &= g(C) + g(E_q) + \frac{3}{2}g(E_q) - \frac{5}{4} \\ &= g(C) + g(F) + \frac{3}{2}g(F) - \frac{5}{4} \\ &\geq q(X) + \frac{3}{2}g(F) - \frac{5}{4}, \end{aligned}$$

where F is a general fiber of h. Since $\kappa(X) = 2$, $g(F) \geq 2$. Hence $g(L) \geq q(X) + \frac{7}{4}$. This case cannot occur. Therefore $g(L) \geq q(X) + 1$. □

3. Some properties of (X, L) with $\kappa(X) = 2$, $h^0(L) > 0$, and $g(L) = q(X)$

Lemma 3.1. *Let X be a smooth projective surface over $\mathbb{C}$.*

(1) *If D is a 1-connected divisor on X, then $h^0(\mathcal{O}_D) = 1$ and $g(D) = h^1(\mathcal{O}_D)$.*

(2) *Let $D = \sum_i a_i D_i$ be an effective divisor on X. If the intersection matrix $\|(D_i \cdot D_j)\|$ is not negative semidefinite, then $h^1(\mathcal{O}_D) \geq q(X)$.*

Proof. First part of (1) is proved by Ramanujam (see Lemma 3 in [Ra]). Last part of (1) is the following. There is an exact sequence

$$0 \to \mathcal{O}_X(-D) \to \mathcal{O}_X \to \mathcal{O}_D \to 0.$$

Hence $\chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D))$. By the Riemann-Roch Theorem, we obtain $\chi(\mathcal{O}_X(-D)) = \chi(\mathcal{O}_X) + \frac{1}{2}(D^2 + DK_X)$. So we have $1 - h^1(\mathcal{O}_D) = \chi(\mathcal{O}_D) = \chi(\mathcal{O}_X) - \chi(\mathcal{O}_X(-D)) = -\frac{1}{2}(D^2 + DK_X)$. Therefore $g(D) = h^1(\mathcal{O}_D)$.

(2) If $q(X) = 0$, then $h^1(\mathcal{O}_D) \geq q(X)$. So we may assume that $q(X) \geq 1$. Let $\alpha(D) = \dim \operatorname{Ker}(H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_D))$.

(A) The case in which $\alpha(D) = q(X)$.

Let $a : X \to \operatorname{Alb}(X)$ be the Albanese map of X. Then $a(D)$ is a point by Lemma 1.3. But this is a contradiction because the intersection matrix of D is not negative semidefinite.

(B) The case in which $0 < \alpha(D) < q(X)$.

Then by Lemma 1.3, there is an Abelian variety G with $\dim G > 0$ and a morphism $f : X \to G$ such that $f(X)$ is not a point and $f(D)$ is a point. But this case cannot occur by the same reason as the case (A). Therefore $\alpha(D) = 0$. Hence $h^1(\mathcal{O}_D) \geq q(X)$. □

Here we study (X, L) under the following assumption.

Assumption A. (X, L): an L-minimal quasi-polarized surface with $\kappa(X) = 2$, $h^0(L) > 0$, and $g(L) = q(X)$. $D = \sum_i a_i C_i$: an effective divisor which is linearly equivalent to L.

We remark that $q(X) \geq 2$ if (X, L) satisfy the Assumption A.

Proposition 3.2. *Under the Assumption* A, *D is a reduced divisor.*

Proof. Let $D_{\text{red}} = \sum_i C_i$ and $D' = D - D_{\text{red}}$. Then

$$g(D) = g(D_{\text{red}}) + \frac{1}{2}(K_X + D + D_{\text{red}})D'.$$

Since D is connected, D_{red} is 1-connected. Because $D^2 > 0$, $g(D_{\text{red}}) = h^1(\mathcal{O}_{D_{\text{red}}}) \geq q(X)$ by Lemma 3.1. We remark that $K_X + D$ is nef. If $D' \neq 0$, then $D'D_{\text{red}} > 0$ by Lemma 1.8. Hence $g(D) \geq q(X) + 1$. This is a contradiction. Therefore $D' = 0$. That is, D is a reduced divisor. □

Proposition 3.3. *Suppose that (X, L) and D satisfy the Assumption* A. *Let $\mu : X' \to X$ be a blowing up at $x \in \bigcup C_i$ and D' the strict transform of D and we put $D' = \mu^* D - aE$, where E is a (-1)-curve such that $\mu(E) = x$. Then $a \leq 2$.*

Proof. We assume that $a > 2$. By the same argument as the proof of Theorem 2.1, we have $\operatorname{Supp} D \subset \operatorname{Bs}|K_X + L|$. Let $M = \frac{a-2}{a}\mu^* D$. Since D is reduced and $\frac{a-2}{a}\mu^* D = \frac{a-2}{a}D' + (a-2)E$, we have $K_{X'} + \lceil M \rceil = K_{X'} + \mu^* D - 2E$. Because M is a nef and big $\mathbb{Q}$-divisor, we obtain

$$H^1(K_{X'} + \mu^* D - 2E) = H^1(K_{X'} + \lceil M \rceil) = 0$$

by the Kawamata-Viehweg vanishing theorem (see Theorem 5.1 in [Sa]). On the other hand

$$H^1(K_{X'} + \mu^* D - 2E) = H^1(\mathcal{O}(K_X + L) \otimes I_x),$$

where I_x is the ideal sheaf of $\{x\}$. (See Lemma 5.1 in [Lz].) So $x \notin \operatorname{Bs}|K_X + L|$. But this is a contradiction because $x \in \operatorname{Supp} D$. Hence $a \leq 2$. □

Corollary 3.4. *Under the Assumption* A,

(a) *The multiplicity of any point of each C_i is at most 2.*
(b) *At $x \in C_i \cap C_j$, C_i and C_j are smooth.*
(c) *$C_i \cap C_j \cap C_k = \varnothing$ for distinct C_i, C_j, C_k.*

Proof. By Proposition 3.3, this is obvious. □

Lemma 3.5 (disconnectedness lemma). *We suppose that (X, L) and D satisfy the Assumption A. Let $x \in \bigcup_i C_i$ and $\mu : X' \to X$ be the blowing up at x and E a (-1)-curve such that $\mu(E) = x$. Let $D' = \sum_i C'_i$ be the strict transform of D and $D' = \mu^* D - aE$. If $a = 2$, then D' is disconnected. In particular D is not irreducible. Moreover $\{x\} = C_i \cap C_j$ for some distinct i, j.*

Proof. Assume that D' is connected. Since D' is reduced, D' is 1-connected. Hence $g(D') = h^1(\mathcal{O}_{D'})$. Next let $\alpha(D') = \dim \operatorname{Ker}(H^1(\mathcal{O}_{X'}) \to H^1(\mathcal{O}_{D'}))$.

(A) The case in which $\alpha(D') = q(X)$.

Let $a : X \to \operatorname{Alb}(X)$ be the Albanese map of X. Then $a(D')$ is a point by Lemma 1.3. On the other hand, $a(E)$ is a point because E is rational and $\operatorname{Alb}(X)$ is an Abelian variety. Therefore $a(D' + E)$ is a point. But since $(D' + 2E)^2 = (\mu^* D)^2 > 0$, this is impossible.

(B) The case in which $0 < \alpha(D') < q(X)$.

Then by Lemma 1.3, there are an Abelian variety G with $\dim G > 0$ and a morphism $f : X \to G$ such that $f(X)$ is not a point and $f(D')$ is a point. Since E is a rational curve, $f(E)$ is a point because G is an Abelian variety. Hence $f(D'+E)$ is a point. But since $(D'+2E)^2 = (\mu^* D)^2 > 0$, this case cannot occur.

Therefore $\alpha(D') = 0$ and $h^1(\mathcal{O}_{D'}) \geq q(X)$. So we have $g(D') \geq q(X)$. But $g(D') = g(D) - 1 = q(X) - 1$ and this is a contradiction. Hence D' is disconnected. In particular D is not irreducible.

Since D' is disconnected, we have $\{x\} \subseteq C_i \cap C_j$ for some distinct i, j. We remark that $x \notin C_k$ for $k \neq i, j$ by Corollary 3.4 (c). If $\sharp\{C_i \cap C_j\} \geq 2$, then D' is connected. Hence $\{x\} = C_i \cap C_j$. □

Definition 3.6. We say that an effective divisor D has a loop $\{C_1, C_2, \ldots, C_r\}$ if there are irreducible reduced curves C_1, C_2, …, C_r ($r \geq 2$) of components of D such that one of the following conditions holds.

1. If $r = 2$, then $\sharp\{C_1 \cap C_2\} \geq 2$.
2. If $r \geq 3$, then $C_i \cap C_{i+1} \neq \varnothing$ for $i = 1, 2, \ldots, r-1$ and $C_r \cap C_1 \neq \varnothing$.

Corollary 3.7. *Under the Assumption* A*,*

(1) *Each C_i is smooth.*
(2) *$C_iC_j = 1$ if $C_i \cap C_j \neq \varnothing$.*
(3) *D has no loops.*

Proof. By Corollary 3.4 and Lemma 3.5, we can easily prove them. □

4. Classification of (X, L) with $\kappa(X) = 2$, $h^0(L) > 0$, and $g(L) = q(X)$

First we prove the following proposition.

Proposition 4.1. *Let (X, L) be an L-minimal quasi-polarized surface with $\kappa(X) = 2$ and $h^0(L) = 1$. Let D be the effective divisor which is linearly equivalent to L. Assume that D is reduced and D has m components. If D satisfies one of the following conditions, then $g(L) \geq q(X) + 1$.*

(1) *For some natural number r with $1 \leq r \leq m-1$, there exist irreducible reduced curves C_1, …, C_r which are components of D such that $\bigcup_{i=1}^{r} C_i$ is connected and*
$$(\sum_{i=1}^{r} a_i C_i)^2 > 0$$
for $a_i \in \mathbb{Z} \setminus \{0\}$ $(i = 1, \ldots, r)$.

(2) *For some natural number r with $1 \leq r \leq m-2$, there exist irreducible reduced curves C_1, …, C_r which are components of D such that $\bigcup_{i=1}^{r} C_i$ is connected and*
$$(\sum_{i=1}^{r} b_i C_i)^2 \geq 0$$
for $b_i \in \mathbb{Z} \setminus \{0\}$ $(i = 1, \ldots, r)$.

Proof. *Case* (1). Let $C_{r+1}, \ldots, C_{m-1}$ be components of D other than $C_1, \ldots, C_r$ such that $\bigcup_{i=1}^{m-1} C_i$ is connected. In this case, there are natural numbers d_r, a_{r+1}, $\ldots$, a_{m-1} such that

$$(d_r(\sum_{i=1}^{r} a_i C_i) + \sum_{i=r+1}^{m-1} a_i C_i)^2 > 0.$$

So we have

$$g(\sum_{i=1}^{m-1} C_i) \geq q(X)$$

by Lemma 3.1. Let

$$D = \sum_{i=1}^{m} C_i.$$

Then

$$\begin{aligned} g(D) &= 1 + \frac{1}{2}(K_X + \sum_{i=1}^{m} C_i)D \\ &= g(\sum_{i=1}^{m-1} C_i) + \frac{1}{2}(K_X + D + \sum_{i=1}^{m-1} C_i)C_m. \end{aligned}$$

On the other hand

$$(K_X + D + \sum_{i=1}^{m-1} C_i)C_m > 0$$

since $K_X + D$ is nef and

$$(\sum_{i=1}^{m-1} C_i)C_m > 0.$$

Therefore $g(L) = g(D) \geq q(X) + 1$.

Case (2). Let $C_{r+1}, \ldots, C_{m-1}$ be components of D other than $C_1, \ldots, C_r$ such that $\bigcup_{i=1}^{m-1} C_i$ is connected. In this case, there are natural numbers d_r, b_{r+1}, $\ldots$, b_{m-1} such that

$$(d_r(\sum_{i=1}^{r} b_i C_i) + \sum_{i=r+1}^{m-1} b_i C_i)^2 > 0.$$

Therefore $g(D) \geq q(X) + 1$ by the same argument as in the case (1). □

Theorem 4.2. *Let (X, L) be a polarized surface with $\kappa(X) = 2$, $h^0(L) > 0$. If $g(L) = q(X)$, then $h^0(L) = 1$ and $1 \leq L^2 \leq 4$. Let D be the effective divisor which is linearly equivalent to L. Then D is a reduced divisor and is one of the following types.*

(1) *D is an irreducible smooth curve.*
(2) *$X \cong C_1 \times C_2$ and $D = F_1 + F_2$, where F_i is a fiber of the projection $X \to C_i$ for $i = 1, 2$. In particular $L^2 = 2$ in this case.*

Proof. We remark that L is ample. By Proposition 3.2, D is a reduced divisor. Since $1 \leq L^2 = D^2 \leq 4$ by Theorem 2.1, D has at most 4 components. By Corollary 3.7 (2), $C_i C_j \leq 1$ for distinct components C_i, C_j of D and each component of D is smooth by Corollary 3.7 (1).

Claim 4.3. *The number of irreducible components of D is smaller than* 3.

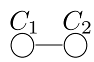
Proof. Assume that D has at least 3 components. By the results in §2 and §3, D has at least one tip curve C_1. By hypothesis, $1 \leq DC_1$. Hence $C_1^2 \geq 0$ by Corollary 3.7 (2). But this is impossible because of Proposition 4.1 (2). □

Therefore the number of irreducible components of D is 1 or 2.

(1) The case in which D has 2 components.

Let $D = C_1 + C_2$. Then the dual graph $G(D)$ of D is the following type.

(3-1)



Claim 4.4. $D^2 = 2$.

Proof. Assume that $3 \leq D^2 \leq 4$. Then we may assume $DC_1 \geq 2$. Hence we have $1 \leq C_1^2$ by Corollary 3.7 (2). But this is impossible because of Proposition 4.1 (1). □

Hence $DC_1 = DC_2 = 1$. So $C_1^2 = C_2^2 = 0$ by Corollary 3.7 (2). Then we prove the following claim.

Claim 4.5. *X is minimal.*

Proof. We remark that $q(X) = g(C_1) + g(C_2)$ and $q(X) \geq 1$. Let $\alpha(C_1) = \dim \operatorname{Ker}(H^1(\mathcal{O}_X) \to H^1(\mathcal{O}_{C_1}))$. If $\alpha(C_1) = 0$, then $q(X) \leq h^1(\mathcal{O}_{C_1}) = g(C_1)$. Hence $q(X) + g(C_2) \leq g(C_1) + g(C_2) = q(X)$. So $g(C_2) = 0$. Since $C_2^2 = 0$, $K_X C_2 < 0$. But this cannot occur since $\kappa(X) = 2$. Therefore $\alpha(C_1) \neq 0$.

If $\alpha(C_1) = q(X)$, then by Lemma 1.3 $a(C_1)$ is a point where $a : X \to \operatorname{Alb}(X)$ is the Albanese map of X. Since $C_1^2 = 0$, $a(X)$ is a curve. But this case cannot occur since $g(L) \geq q(X) + 1$ by Theorem 5.5 in [Fk1].

Hence $0 < \alpha(C_1) < q(X)$. Then there is an Abelian variety G with $\dim G > 0$ and a morphism $h_2 : X \to G$ such that $h_2(X)$ is not a point and $h_2(C_1)$ is a point. Since $C_1^2 = 0$, $\dim h_2(X) = 1$. By taking the Stein factorization, we get a fiber space $f_2 : X \to B_2$ where B_2 is a smooth curve with $g(B_2) \geq 1$. Then $F_2 \equiv m_1 C_1$ where F_2 is a general fiber of f_2 and $\equiv$ denotes numerical equivalence.

By the same argument as above for C_2, we can prove that $0 < \alpha(C_2) < q(X)$ and we get a fiber space $f_1 : X \to B_1$ where B_1 is a smooth curve with $g(B_1) \geq 1$. Then $F_1 \equiv m_2 C_2$, where F_1 is a general fiber of f_1. If X is not minimal, then there is a (-1)-curve E on X. Since E is rational and $g(B_i) \geq 1$ for $i = 1,2$, E is contained in a fiber of f_1 and a fiber of f_2. Hence $C_1 E = C_2 E = 0$. But this case cannot occur since $L = C_1 + C_2$ is ample. Therefore X is minimal and this completes the proof of Claim 4.5. □

Next we prove Claim 4.6. (This claim was proved by T. Fujita.)

Claim 4.6 (T. Fujita). $X \cong C_1 \times C_2$.

Proof. By Lefschetz's theorem, $H_1(D,\mathbb{Z}) \to H_1(X,\mathbb{Z})$ is surjective. Since C_1 and C_2 intersect transversally, $\operatorname{rank} H_1(D,\mathbb{Z}) = \operatorname{rank} H_1(C_1,\mathbb{Z}) + \operatorname{rank} H_1(C_2,\mathbb{Z})$. We remark that $\operatorname{rank} H_1(C_i,\mathbb{Z}) = 2g(C_i)$ for i =1, 2. Hence

$$\begin{aligned} 2q(X) &= \operatorname{rank} H_1(X,\mathbb{Z}) \\ &\leq \operatorname{rank} H_1(D,\mathbb{Z}) \\ &= \operatorname{rank} H_1(C_1,\mathbb{Z}) + \operatorname{rank} H_1(C_2,\mathbb{Z}) \\ &= 2g(C_1) + 2g(C_2). \end{aligned}$$

Since $q(X) = g(C_1) + g(C_2)$, we have $H_1(X,\mathbb{Z})/\operatorname{Tor} \cong H_1(C_1,\mathbb{Z}) \oplus H_1(C_2,\mathbb{Z})$. Let $r_i : H^0(X,\Omega^1_X) \to H^0(C_i,\Omega^1_{C_i})$ for i =1, 2 and $r = r_1 \oplus r_2$. Since $\operatorname{Ker} r_1 \cap \operatorname{Ker} r_2 = 0$, r is an isomorphism $H^0(X,\Omega^1_X) \cong H^0(C_1,\Omega^1_{C_1}) \oplus H^0(C_2,\Omega^1_{C_2})$. On the other hand, $\operatorname{Alb}(X) \cong H^0(X,\Omega^1_X)^\vee/(H_1(X,\mathbb{Z})/\operatorname{Tor})$ and $J(C_i) \cong H^0(C_i,\Omega^1_{C_i})^\vee/H_1(C_i,\mathbb{Z})$ for i =1, 2, where $\vee$ denote the dual. Hence there is a natural morphism $\varphi : \operatorname{Alb}(X) \to J(C_1)$ by the above argument. Let $f = \varphi \circ \alpha$ where $\alpha : X \to \operatorname{Alb}(X)$. Then $f(C_2)$ is a point by the definition of f and $f|_{C_1}$ is the Albanese map of C_1. Because $C_2^2 = 0$, $f(X)$ is a curve. Therefore $f(X) \cong C_1$ and $mC_2 = f^{-1} \circ f(C_2)$ for some $m \in \mathbb{N}$. We remark that $f|_{C_1} : C_1 \to f(C_1)$ is an isomorphism and $f(C_1) = f(X)$. Therefore there is a morphism $f : X \to f(X) = C_1$ such that f has a section C_1. Hence C_2 is a fiber of f, that is, $f^{-1}(x) = C_2$ for some $x \in C_1$. Let F be a general fiber of f. Since $q(X) = g(C_1) + g(C_2) = g(C_1) + g(F)$ and X is minimal, $X \cong C_1 \times C_2$ by Beauville's result ([Be]). □

(2) The case in which D has one component.

Then D is an irreducible reduced smooth curve by Proposition 3.2 and Corollary 3.7 (1). We complete the proof of Theorem 4.2. □

5. Example and problem

Example 5.1. (See [Ln].) Let C be a smooth curve with $g(C) \geq 3$. Let $S^2(C)$ be the 2-fold symmetric product of C. Then $S^2(C)$ is of general type. Let $\pi : C \times C \to S^2(C)$ be the natural map. We put $L = \pi(C \times \{x\})$ and $X = S^2(C)$, where x is a point of C. Then L is an ample irreducible smooth curve with $L^2 = 1$ and $g(L) = q(X)$.

Problem 5.2. Does there exist an example of (X, L) of the type (1) in Theorem 4.2 with $L^2 = 2$, 3 or 4?

References

[Be] A. Beauville, *L'inegalite $p_g \geq 2q - 4$ pour les surfaces de type général*, Bull. Soc. Math. France **110** (1982), 343–346.

[Bo] E. Bombieri, *Canonical models of surfaces of general type*, Inst. Hautes Etudes Sci. Publ. Math. **42** (1973), 171–219. MR **47:**6710

[Ca] F. Catanese, *On the moduli spaces of surfaces of general type*, J. Diff. Geom. **19** (1984), 483–515. MR **86h:**14031

[De] J.-P.Demailly, *Effective bounds for very ample line bundles*, Invent Math. **124** (1996), 243–261. CMP 96:06

[Fj1] T. Fujita, *Classification Theories of Polarized Varieties*, London Math. Soc. Lecture Note Series **155** (1990). MR **93e:**14009

[Fj2] ———, *On the structure of polarized varieties with Δ-genera zero*, J. Fac. Sci. Univ. of Tokyo **22** (1975), 103–115. MR **51:**5596

[Fk1] Y. Fukuma, *A lower bound for the sectional genus of quasi-polarized surfaces*, to appear in Geometriae Dedicata.

[Fk2] ———, *A lower bound for sectional genus of quasi-polarized manifolds*, to appear in J. Math. Soc. Japan.
[Ln] A. Lanteri, *Algebraic surfaces containing a smooth curve of genus $q(S)$ as an ample divisor*, Geometriae Dedicata **17** (1984), 189–197. MR **86d:**14031
[Lz] R. Lazarsfeld, *Lectures on Linear Series*, preprint.
[Mo] S. Mori, *Classification of higher dimensional varieties*, Proc. Symp. in Pure Math **46** (1987), 269–331. MR **89a:**14040
[Ra] C. P. Ramanujam, *Remarks on the Kodaira vanishing theorem*, J. Indian Math. Soc. **36** (1972), 41–51. MR **48:**8502
[Re] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. Math. **127** (1988), 309–316. MR **89e:**14038
[Sa] F. Sakai, *Weil divisors on normal surfaces*, Duke Math. J. **51** (1984), 877–887. MR **86m:**14025
[So] A. J. Sommese, *On the adjunction theoretic structure of projective varieties*, Lecture Notes in Mathematics **1194** (1986), 175–213. MR **87m:**14049
[SV] A. J. Sommese and A. Van de Ven, *On the adjunction mapping*, Math. Ann. **278** (1987), 593–603. MR **88j:**14011

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